LOWER BOUND ON THE PION POLARIZABILITY FROM QCD SUM RULES

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(February 1, 2008)

Abstract

Making use of QCD sum rules a lower bound is found which relates the electromagnetic polarizability $\alpha_{_E}$ and mean-square radius $\langle r_\pi^2\rangle$ of charged pions through the intrinsic polarizability $\tilde{\alpha}_{_E} = \alpha_{_E} - \alpha\langle r_\pi^2\rangle/(3M_\pi)$. We find that if present constraints on the QCD continuum (duality) threshold are accepted, this lower bound on the intrinsic polarizability $\tilde{\alpha}_{_E}$ is incompatible with some previous determinations of $\alpha_{_E}$ and $\langle r_\pi^2\rangle$.

Recently the electromagnetic polarizability of charged pions has been the object of several theoretical and experimental investigations, since it is believed that one can make direct predictions based upon quantum chromodynamics (QCD).

In the standard unit of $10^{-4}\,\mathrm{fm}^3$ (which will be used throughout this paper), chiral perturbation theory predicts $\alpha_{_E} = 2.8$ [1,2], while alternative approaches based on the Das-Mathur-Okubo (DMO) sum rule [3] have resulted in values ranging from 2.6 (based on resonance saturation [1]) to 5.6 ± 0.5 [4]. Other theoretical calculations in quark and NJL

models range between 3.6 and 13 [5]. On the experimental side there are three measurements of $\alpha_{_E}$, ranging from $2.2 \pm\, 1.6$ [6] to $6.8 \pm\, 1.4\, \pm 1.2$ [7], and to $20 \pm\, 12$ [8].

In view of the large experimental and theoretical uncertainties it is worthwhile to search for theoretical bounds related to $\alpha_{_E}$. This is done in this work using the relation between the polarizability and the axial current form factor in the decay $\pi \to\, l\, \nu\, \gamma$ [9] and the above mentioned current algebra sum rule [3] which states that the intrinsic polarizability $\tilde{\alpha}_{_E}$ is given by

$$\tilde{\alpha}_{_E} = \alpha_{_E} - \frac{\alpha}{3M_\pi} < r_\pi^2 >= \frac{\alpha}{M_\pi\, f_\pi^2} \int\limits_{4M_\pi^2}^{\infty} dt \frac{1}{t}\, [\rho^A(t) - \rho^V(t)]\, , \tag{1}$$

where M_π is the pion mass, $\langle r_\pi^2 \rangle$ is the mean-square pion radius, and $f_\pi = 133\,\mathrm{MeV}$ is the pion decay constant. In the derivation of the DMO sum-rule, the pion pole contribution is explicitly removed from the right-hand side in (1) as reflected by the $4\, M_\pi^2$ threshold. The value of the mean-square radius is also uncertain, with values ranging from $\langle r_\pi^2 \rangle = 6/M_\rho^2 = 0.4\,\mathrm{fm}^2$ from vector meson dominance, to the experimental results $\langle r_\pi^2 \rangle = (0.439 \pm 0.008)\,\mathrm{fm}^2$ [10] and $\langle r_\pi^2 \rangle = (0.463 \pm 0.006)\,\mathrm{fm}^2$ [11].

The spectral functions $\rho^A(t)$ and $\rho^V(t)$ are given by the absorptive parts of the axial and vector current correlators, respectively:

$$\rho^{A,V}(t) = \frac{1}{\pi} Im\, \Pi^{A,V}(q^2)\, , \tag{2}$$

where

$$\Pi^{A,V}(q^2) = \frac{4}{3}\, i \left(\frac{q_\mu q_\nu}{q^2} - \frac{1}{4} g_{\mu\nu} \right) \int d^4x e^{iq\cdot x} < 0|T(J_\mu^{A,V}(x) J_\nu^{A,V}(0))|0 >\, , \tag{3}$$

with

$$J_\mu^V(x) = \bar{u}\gamma_\mu d\, , \qquad\qquad J_\mu^A(x) = \bar{u}\gamma_\mu\gamma_5 d \quad . \tag{4}$$

Of course large differences in the value of the intrinsic polarizability $\tilde{\alpha}_{_E}$ arise because the sum rule involves cancellation of large numbers between the vector and axial spectral functions. To avoid this problem, we will show that for each term of the sum rule one can

find interesting upper and/or lower bounds based on QCD sum rules. These bounds are a function QCD condensates and of the continuum (duality) thresholds. A suitable combination of these bounds results in a meaningful lower bound on the intrinsic polarizability $\tilde{\alpha}_{_E}$ relating $\alpha_{_E}$ and $\langle r_\pi^2 \rangle$.

Our main aim is to find a lower bound to the integral in (1)

$$I = \int_{4M_\pi^2}^{\infty} dt \frac{1}{t} \left[\rho^A(t) - \rho^V(t) \right] . \tag{5}$$

This can be rewritten as

$$I = I^A \left(s_0^A \right) - I^V \left(s_0^V \right) + C^{AV} \left(s_0^A, s_0^V \right) , \tag{6}$$

where

$$I^{A,V} \left(s_0^{A,V} \right) = \int_{4M_\pi^2}^{s_0^{A,V}} dt \frac{1}{t} \, \rho^{A,V}(t) \tag{7}$$

$$C^{AV} \left(s_0^A, s_0^V \right) = \int_{s_0^A}^{s_0^V} dt \frac{1}{t} \, \tilde{\rho}(t) . \tag{8}$$

In the above equations, $s_0^{A,V}$ is the continuum threshold for duality with asymptotic freedom in the axial and vector channels, *i.e.* for $t \geq s_0^{A,V}$ the spectral function behaves as predicted by perturbative QCD. The spectral function denoted by $\tilde{\rho}(t)$ represents its perturbative expression which is the same for the axial and vector case for light quarks up to two-loop order.

Since the integrand of I^A is positive definite the following inequality holds [12] (see [13] and [14] for further applications of inequalities and QCD sum-rules)

$$I^A \left(s_0^A \right) \geq L^A \left(s_0^A \right) = \frac{F}{(1 - D^2/E)} , \tag{9}$$

where F, D, E are combinations of integrals *related to* the finite-energy sum rules (the reason for this distinction will be seen below)

$$m_n^{A,V} = \int_{4M_\pi^2}^{s_0^{A,V}} dt \, t^n \, \rho^{A,V}(t) \tag{10}$$

and in particular

$$F = \frac{m_0^2}{m_1}\,, \tag{11}$$
$$D = \frac{m_2}{m_1} - \frac{m_1}{m_0} \tag{12}$$
$$E = \frac{m_3}{m_1} - 2\frac{m_2}{m_0} + (\frac{m_1}{m_0})^2\,. \tag{13}$$

The explicit results for the FESR in the axial and vector channels are given by [15]

$$\mathcal{F}_n^{A,V} = \int_0^{s_0^{A,V}} t^n Im\Pi^{A,V}(t)\, dt \tag{14}$$
$$\mathcal{F}_n^{A,V} = \frac{1}{4\pi^2}\left[\frac{(s_0^{A,V})^{(n+1)}}{(n+1)}\left(1 + F_{2n+2}(s_0^{A,V})\right) + (-1)^n C_{2n+2} < O_{2n+2} >_{A,V}\right]\,, \tag{15}$$

where $F_p(s_0)$ are the radiative corrections, which, for three colours and three flavours are given by

$$F_p(s_0) = \frac{\alpha_S(s_0)}{\pi} + \left[\frac{\alpha_S(s_0)}{\pi}\right]^2\left(1.641 + \frac{9}{2\,p} - \frac{16}{9}\ \ln\ \ln\frac{s_0}{\Lambda_{QCD}^2}\right) \tag{16}$$

and $C_{2n+2} < O_{2n+2} >_{A,V}$ contain the quark and gluon condensates [16]. Separating out the pion pole contribution from $Im\Pi^A(t)$ in (14) as required by the DMO sum-rule (1) gives [1]

$$\mathcal{F}_n^A = f_\pi^2 M_\pi^{2n} + \int_{4M_\pi^2}^{s_0^A} t^n Im\Pi^A(t)dt \tag{17}$$
$$m_n^A = \mathcal{F}_n^A - f_\pi^2 M_\pi^{2n} \quad . \tag{18}$$

Of course in the vector channel m_n^V and $\mathcal{F}_n^V$ are identical.

An upper bound for $I^V\left(s_0^V\right)$ can be found using the following relation based again on Schwartz inequalities

$$m_{-2} \geq \frac{FF}{(1 - DD^2/EE)}\,, \tag{19}$$

where FF, DD and EE are obtained from F, D and E of (12-13) replacing m_n by m_{n-1}. The previous inequality is equivalent to

[1]We are grateful to the referee for bringing this point to our attention.

$$I^V\left(s_0^V\right) \leq \frac{1}{m_2}\left(m_0 m_1 + \sqrt{m_0^2 m_1^2 - m_0^3 m_2 + m_{-2} m_2 (m_0 m_2 - m_1^2)}\right) . \tag{20}$$

Since m_{-2} depends on the extreme infrared properties of QCD a theoretical prediction for this sum-rule does not exist. However, by recognizing that the currents $J_\mu^V(x) = \bar{u}\gamma_\mu u$ and $J_\mu^V(x) = \bar{u}\gamma_\mu d$ lead to identical correlation functions in the $SU(2)$ limit, an upper bound on m_{-2} can be found in terms of the hadronic contributions to the anomalous magnetic moment of the muon. In fact one has the following series of inequalities [17–19]

$$\begin{aligned} a_\mu^{had} &= 4\alpha^2 \sum_f Q_f^2 \int_{4M_\pi^2}^{\infty} dt\, K_\mu(t)\, \rho^V(t) \geq \frac{4}{3}\alpha^2 \sum_f Q_f^2 M_\mu^2 \int_{4M_\pi^2}^{\infty} dt\, \frac{1}{t^2}\, \rho^V(t) \\ &\geq \frac{4}{3}\alpha^2 \sum_f Q_f^2 M_\mu^2 \int_{4M_\pi^2}^{s_0^V} dt\, \frac{1}{t^2}\, \rho^V(t) = \frac{20}{27}\alpha^2 M_\mu^2\, m_{-2} \, . \end{aligned} \tag{21}$$

namely

$$m_{-2} \leq \frac{27}{20}\frac{a_\mu^{had}}{\alpha^2 M_\mu^2} \tag{22}$$

and finally

$$I^V\left(s_0^V\right) \leq U^V\left(s_0^V\right) = \frac{1}{m_2}\left(m_0 m_1 + \sqrt{m_0^2 m_1^2 - m_0^3 m_2 + \frac{27}{20}\frac{a_\mu^{had}}{\alpha^2 M_\mu^2} m_2 (m_0 m_2 - m_1^2)}\right) . \tag{23}$$

Combining equations (6), (7), (9) and (22) one has

$$I \geq I^L\left(s_0^A, s_0^V\right) = L^A\left(s_0^V\right) - U^V\left(s_0^V\right) + C^{AV}\left(s_0^A, s_0^V\right) \tag{24}$$

and

$$\tilde{\alpha}_E \geq \frac{\alpha}{M_\pi f_\pi^2} I^L\left(s_0^A, s_0^V\right) . \tag{25}$$

In Figures 1 and 2 the quantity $\alpha I^L\left(s_0^A, s_0^V\right)/(M_\pi f_\pi^2)$ is plotted as a function of s_0^A and s_0^V. Since only u and d flavours are used in the currents, the value of a_μ^{had} has been taken equal to 6×10^{-8} [19,20], a value which includes contributions from the light (u, d) resonances only (strange quarks are a 10% effect). As to the condensates, both standard values [21] and values from [22] have been used in the FESR. The figures show that a negative value of $\tilde{\alpha}_E$ as found in the resonance saturation approach [1] is consistent with our bounds.

The values of the QCD continuum that have been determined from sum-rule applications range from $1.5\,\mathrm{GeV}^2 < s_0^V < 4.0\,\mathrm{GeV}^2$ for the vector channel [21–24] and $1.75\,\mathrm{GeV}^2 < s_0^A < 2.5\,\mathrm{GeV}^2$ in the axial channel [22–25]. With these bounds on the continuum threshold, Figures 1 and 2 give a lower bound on $\tilde{\alpha}_E$ that lies between -12.3 and -11.3, reflecting uncertainties in the QCD condensates and continuum thresholds.

The shaded region of Figure 3 shows the region of α_E, $\langle r_\pi^2\rangle$ parameter space consistent with these lower bounds on the intrinsic polarizability. As is evident from the Figure, even the most pessimistic bound (the lowest diagonal line) is in disagreement with the largest experimental value of $\langle r_\pi^2\rangle$ combined with the chiral perturbation theory or resonance saturation prediction of α_E. In other words, the lower values of α_E combined with larger values of $\langle r_\pi^2\rangle$ are incompatible with the QCD determinations of the continuum thresholds representing the minimum energy necessary for duality.

The present analysis seems to point towards larger values of α_E and lower values of $\langle r_\pi^2\rangle$. Further studies of the continuum (duality) thresholds s_0^A and s_0^V, better knowledge of the mean-square pion radius, and further experimental determinations of α_E would be valuable in helping to clarify our understanding of QCD.

Acknowledgements: MB and TGS are grateful for the financial support of the Natural Sciences and Engineering Research Council of Canada (NSERC). Many thanks to J. Kwan for assistance with the preparation of the figures.

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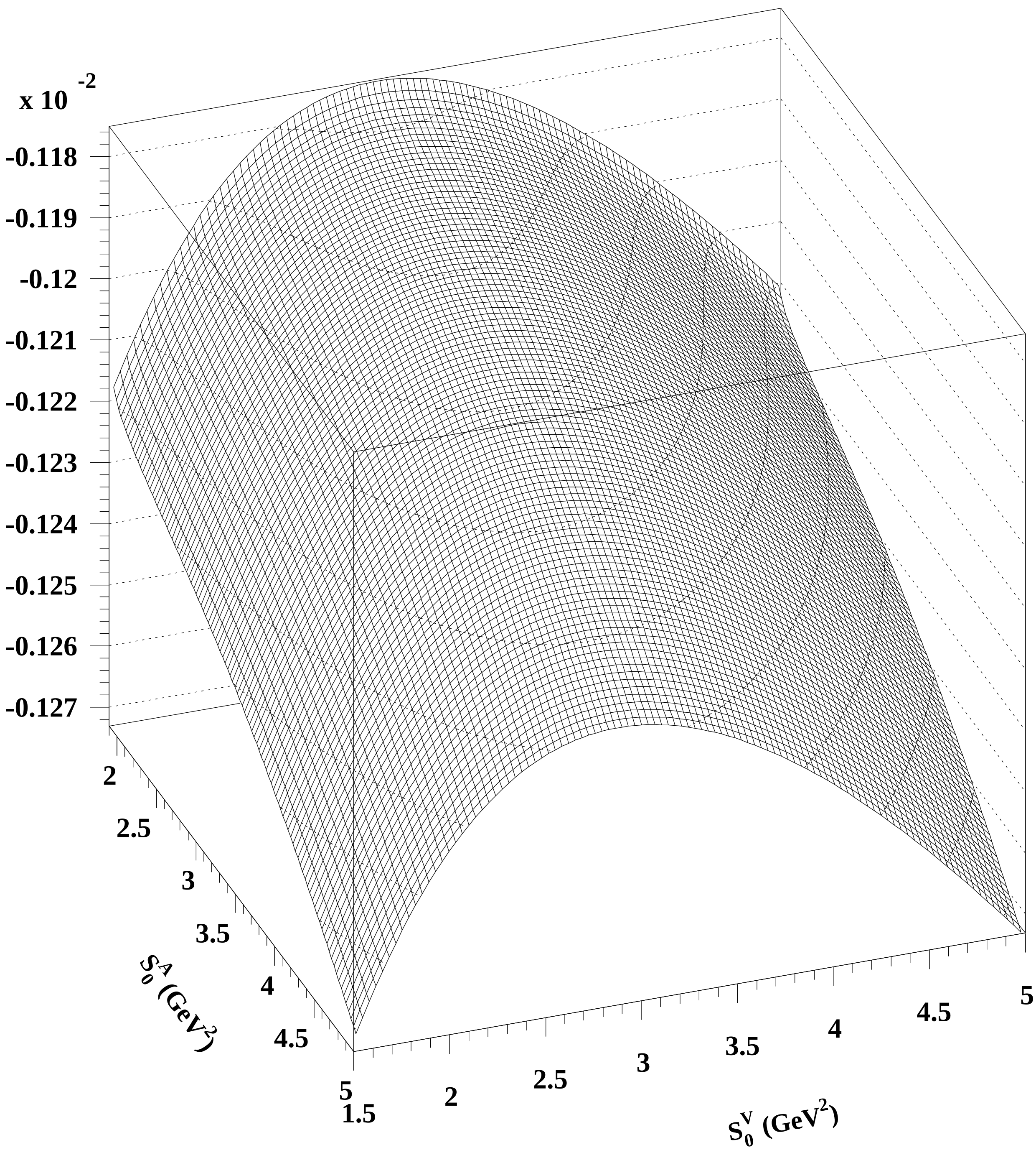
FIGURES

FIG. 1. Plot of $\alpha I^L/(M_\pi f_\pi^2)$ in fm units as a function of s_0^A and s_0^V for standard values [21] of the condensates.

FIG. 2. Same as Figure 1 except for using values of the condensates as determined in [22].

FIG. 3. Shaded region represents the $\alpha_{_E}$, $\langle r_\pi^2\rangle$ parameter space consistent with our QCD sum-rule bound on $\tilde{\alpha}_{_E}$. Diagonal lines represent possible borders of the parameter space for various choices of the condensates and continuum threshold. The horizontal line represents the chiral perturbation theory prediction of $\alpha_{_E}$.

Figure 1



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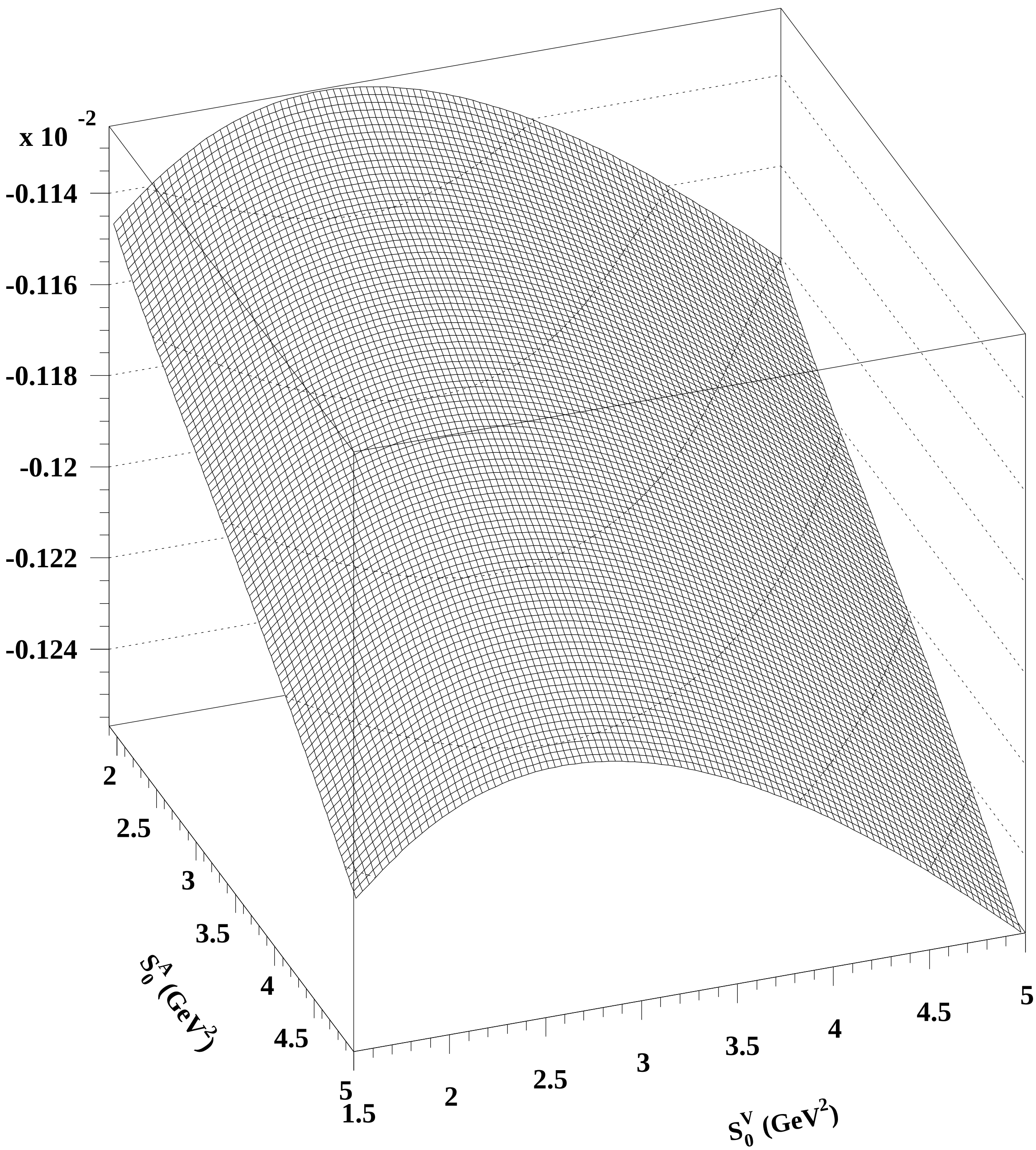
-0.119
-0.12
-0.121
-0.122
-0.123
-0.124
-0.125
-0.126
-0.127
2
2.5
3
3.5
4
4.5
5
S_0^A (GeV^2)
1.5
2
2.5
3
3.5
4
4.5
5
S_0^V (GeV^2)

Figure 2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-0.122
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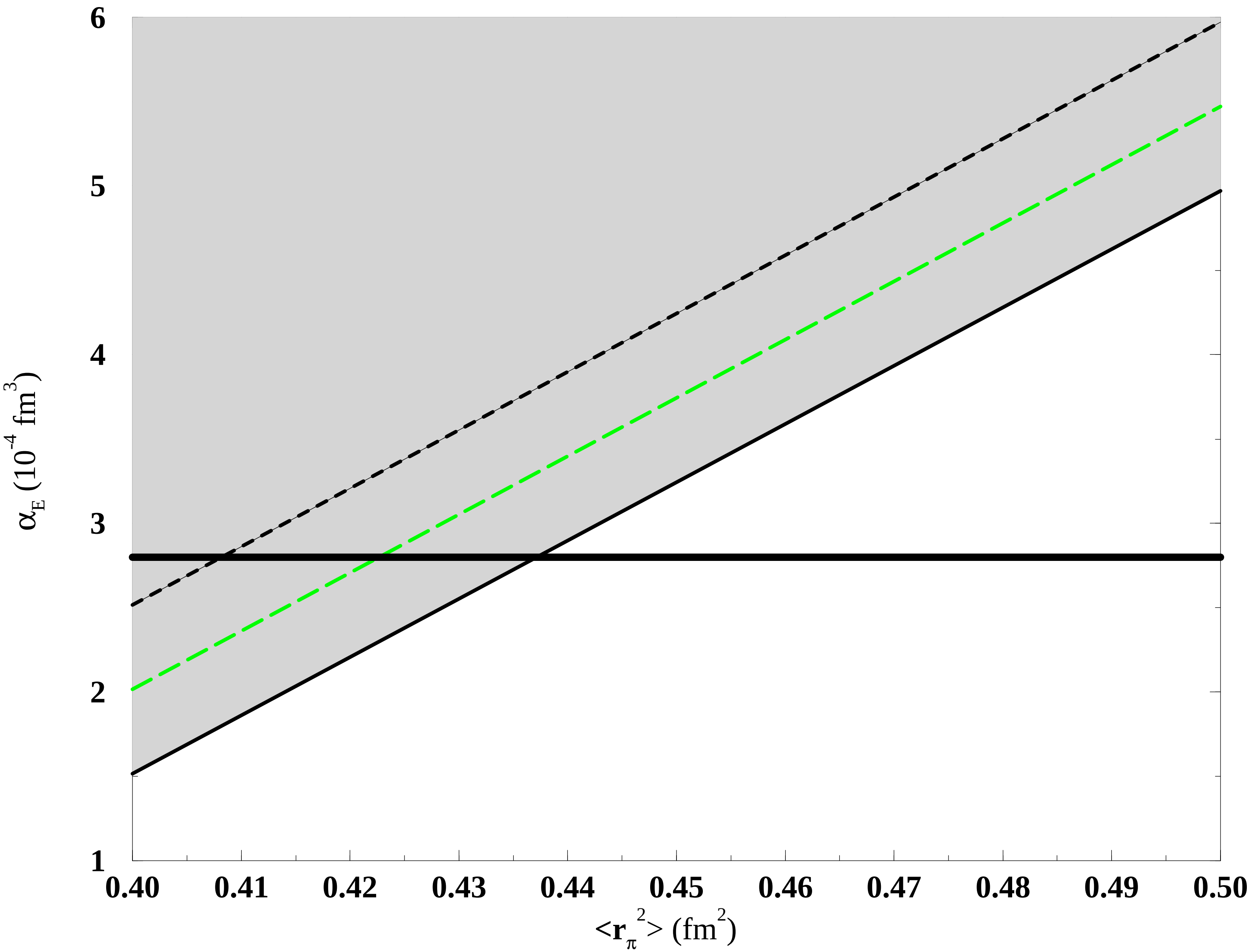
2
2.5
3
3.5
4
4.5
5
S_0^A (GeV^2)
1.5
2
2.5
3
3.5
4
4.5
5
S_0^V (GeV^2)

Figure 3


6
5
4
3
2
1
α_E (10^{-4} fm^3)
0.40
0.41
0.42
0.43
0.44
0.45
0.46
0.47
0.48
0.49
0.50
$\langle r_\pi^2 \rangle$ (fm^2)